Definitions

Definition. The **sample space** is the set of all possible outcomes.

For example:

- For a coin flip, the sample space is:
- For a 6-sided die, the sample space is:
- For poker hand, the sample space is:
- For a PIN, the sample space is:
- For a sequence of 3 coin flips, the sample space is:
- For a sequence of 3 dice rolls where we only record the sum and whether or not each die was even:

Definition. An **event** is a set of outcomes, ie (not necessarily strict) subset of the *sample space*

For example:

- The coin coming up heads is:
- The die rolling an even number is:
- The die rolling a prime is:
- Getting AK is:
- Square number PIN is:
- More heads than tails is:
- Rolling a total of 12 is:

Definition. For discrete sample spaces **probability of an outcome** $\mathbb{P}(o)$ is a real number $\in [0,1]$, and the sum across all possible outcomes is 1.

For example:

- For a fair coin:
- For a fair 6-sided die:
- From a well-shuffled deck and a fair dealer:

Definition. For discrete sample spaces **probability of an event**, E is $\mathbb{P}(E)$ is the sum across all outcomes in E. ie

$$\mathbb{P}(E) = \sum_{o \in E} \mathbb{P}(o)$$

- The probability of a fair coin coming up heads is:
- The probability of a fair die rolling an even number is:
- The probability of a fair die rolling a prime is:
- The probability of getting AK from a fair dealer is:
- The probability of a square number PIN chosen at random with all numbers equally likely is:
- The probability of getting at least two heads is:
- The probability of getting 13 and at exactly one odd die is:

Example

What is the probability after rolling two fair dice the sum is 6?

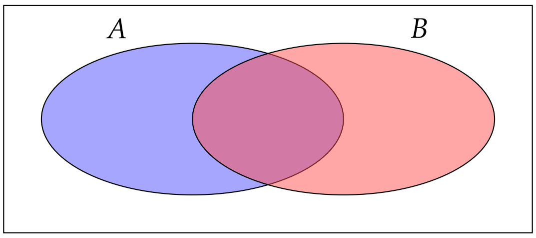
What is the sample space, and the probabilities of each outcome?

Venn Diagrams

Example

Given that $\mathbb{P}(A \cap B') = 0.13, \mathbb{P}(A' \cap B') = 0.44$, what are:

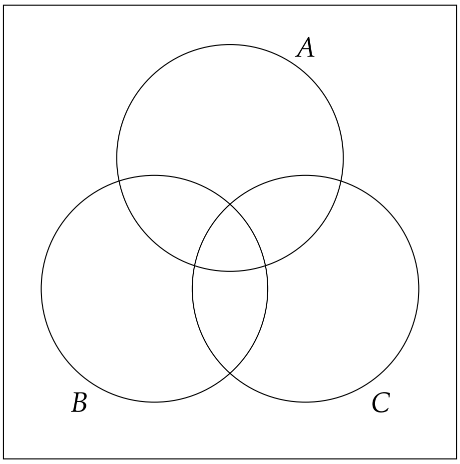
$$\mathbb{P}(B), \quad \mathbb{P}(A \cup B)$$

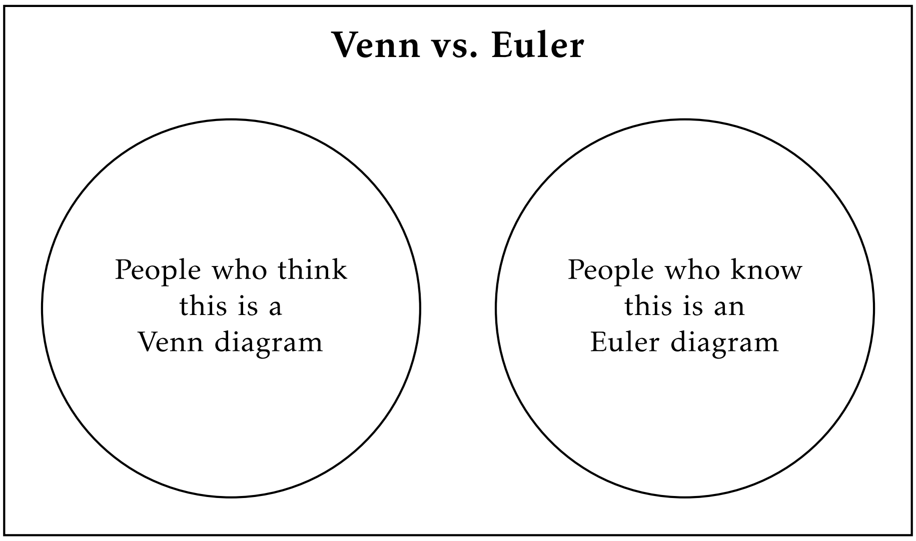


Example

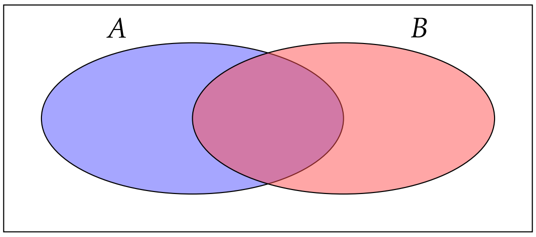
A sample space S is given as $\{1,2,3,4,5,6,7,8,9,10\}$. Three events in this sample space are $A = \{2,4,6,8,10\}, B = \{1,2,3,4,5\}$ and $C = \{4,5,6,8,9\}$.

Find (a) $A \cap B$, (b) $B \cup C$, (c) $A' \cup B$, (d) $A \cap B'$, (e) $(A \cap C)'$.




Venn vs. Euler
People who think this is a Venn diagram
People who know this is an Euler diagram

Addition Law



Fact —

$$\mathbb{P}(A \cup B) =$$

Principle of Inclusion-Exclusion

Fact —

$$\mathbb{P}(A \cup B \cup C) =$$

Example

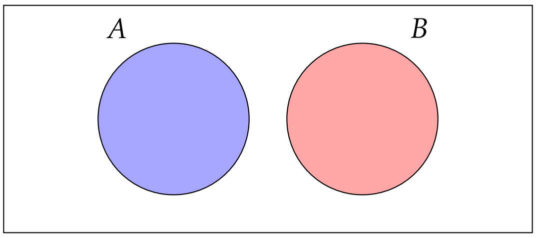
A café serves a basic breakfast of bacon and eggs. Customers may choose any combination of mushrooms, beans and tomatoes as optional extras if they wish. The probability that a customer chooses at least one of these is 0.88. The probability that a customer chooses beans or tomatoes or both is 0.74, beans or mushrooms or both 0.63, tomatoes or mushrooms or both 0.81, tomatoes 0.57, beans 0.40 and mushrooms 0.38. Find the probability that a customer

(a) chooses all three of beans, mushrooms and tomatoes

(b) chooses tomatoes but not beans or mushrooms

Mutually Exclusive Events

Recall that $\mathbb{P}(A \cup B) = \mathbb{P}(A) + \mathbb{P}(B) - \mathbb{P}(A \cap B)$. When $\mathbb{P}(A \cap B) = 0$, this equation becomes: $\mathbb{P}(A \cup B) = \mathbb{P}(A) + \mathbb{P}(B)$, and we say A and B are **mutually exclusive**.



Example

(a) What is the probability a die is 2 or a multiple of 3?

(b) What is the probability of a pair of Aces or a pair of Kings?

MECE - Mutually Exclusive Collectively Exhaustive

Sometimes it is helpful to break the sample space into *mutually exclusive* but also *collectively exhaustive*, ie a set of events which are pairwise disjoint, but cover the whole sample space.

Conditional Probability

Often it is useful to think about what might happen, after a specific event has happened.

Example

The probability you revise is .7, if you revise the probability you get an A^* is .8, if you don't revise, the probability is .6. What is the probability you get an A^*?

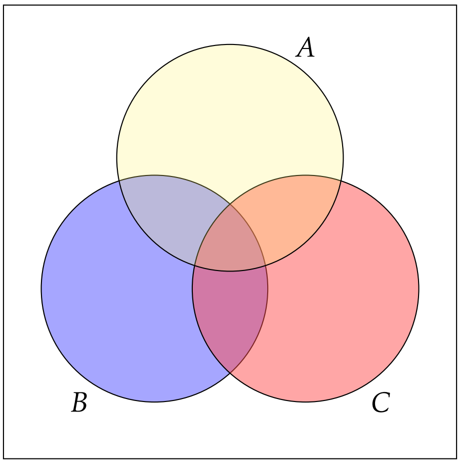
We are using an important principle here:

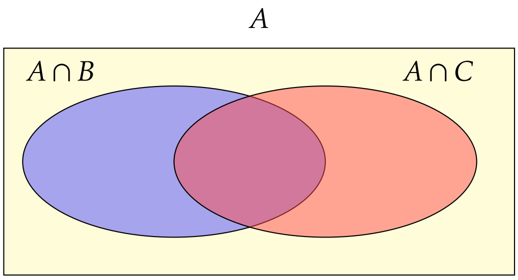
Fact — If A and B are two events, and $\mathbb{P}(A) > 0$, then the **conditional probability** of B given A is

$$\mathbb{P}(B \mid A) = \frac{\mathbb{P}(A \cap B)}{\mathbb{P}(A)}$$

This can be re-written as $\mathbb{P}(A \cap B) = \mathbb{P}(A) \times \mathbb{P}(B \mid A)$. This is the **multiplication law of probability**

It is sometimes helpful to consider $\mathbb{P}(\cdot \mid A)$ as a brand new *probability* on sample space A.



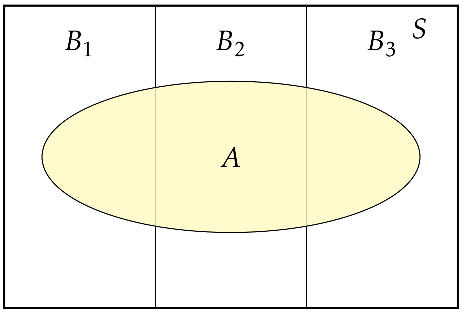


Law of Total Probability

Sometimes we want to find the probability of an event A, but we only know how likely A is to happen under specific conditions (scenarios).

If we have a set of events $B_1, B_2, \dots, B_n$ that are **Mutually Exclusive** (no overlap) and **Collectively Exhaustive** (cover all possibilities), they form a **partition** of the sample space.

> Fact — To find $\mathbb{P}(A)$, we sum the probabilities of A occurring in each scenario:
>
> $$\mathbb{P}(A) = \mathbb{P}(A \cap B_1) + \mathbb{P}(A \cap B_2) + \cdots + \mathbb{P}(A \cap B_n)$$
>
> Using the multiplication law, this is most commonly written as:
>
> $$\mathbb{P}(A) = \mathbb{P}(A \mid B_1)\mathbb{P}(B_1) + \mathbb{P}(A \mid B_2)\mathbb{P}(B_2) + \cdots + \mathbb{P}(A \mid B_n)\mathbb{P}(B_n)$$



Example

A factory has three machines making bolts.

- Machine 1 produces 20% of the bolts and 1% are defective.
- Machine 2 produces 30% of the bolts and 2% are defective.
- Machine 3 produces 50% of the bolts and 3% are defective.

If a bolt is chosen at random, what is the probability it is defective?

Example

You have two bags of marbles.

- Bag A contains 3 Red marbles and 2 Blue marbles.
- Bag B contains 1 Red marble and 4 Blue marbles.

You flip a fair coin. If it is Heads, you pick from Bag A. If it is Tails, you pick from Bag B. What is the probability of picking a Red marble?

Bayes Theorem

Since $\mathbb{P}(A \mid B) = \frac{\mathbb{P}(A \cap B)}{\mathbb{P}(B)}$ and $\mathbb{P}(B \mid A) = \frac{\mathbb{P}(B \cap A)}{\mathbb{P}(A)}$, we can relate $\mathbb{P}(A \mid B)$ to $\mathbb{P}(B \mid A)$ as follows:

Theorem (Bayes Theorem)

$$\mathbb{P}(A \mid B) = \mathbb{P}(A) \frac{\mathbb{P}(B \mid A)}{\mathbb{P}(B)}$$

A useful framing for this is to think of A as some event we're interested in, and B as some piece of evidence that A might be true. Then we are saying, "the probability of A given some data B is the probability of A multiplied by the probability of seeing that data in the world where A is true, divided by the probability of seeing that data."

Another useful framing is $P(A \mid B) \propto \mathbb{P}(A)\mathbb{P}(B \mid A)$

Example

99 people in a room are right-handed, and 1 is left-handed. How many right-handed people need to leave the room for the room to be 98% right-handed?

Example

Suppose a test for a rare cancer has a **sensitivity** of 99.9%, i.e., if someone has cancer, then 999 times out of $1,000$, the test is positive. Suppose also it has a **specificity** of 99%, that is if you don't have the rare cancer, then 99% of the time the test will be negative. Given that 0.1% of the population has this cancer and the test is positive for someone, what is the probability they have cancer?

Independent Events

Definition. Two events A and B are **independent** if knowing that one event has occurred does not affect the probability of the other event occurring.

Mathematically, A and B are independent if and only if:

$$\mathbb{P}(A \cap B) = \mathbb{P}(A) \cdot \mathbb{P}(B)$$

For example:

- When rolling a fair die twice, getting a 6 on the first roll and getting an even number on the second roll are:
- When drawing cards with replacement, getting an ace on the first draw and a heart on the second draw are:
- When drawing cards without replacement, getting an ace on the first draw and a heart on the second draw are:

Example

A bag contains 3 red balls and 4 blue balls. Two balls are drawn from the bag. Are drawing a red ball first and drawing a blue ball second independent events?

Fact — For independent events A and B:

- $\mathbb{P}(A|B) = \mathbb{P}(A)$
- $\mathbb{P}(B|A) = \mathbb{P}(B)$

This provides another way to check if events are independent.

Example

If $\mathbb{P}(A) = 0.3$, $\mathbb{P}(B) = 0.4$, and $\mathbb{P}(A \cap B) = 0.12$, are A and B independent?